Particle Physics II

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Exercise sheet I

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Please use $c = 3 \times 10^8\,\mathrm{m/s}$ and $\hbar c = 0,2\,\mathrm{GeV\,fm}$ for all numerical computations.

In-class exercises

Exercise 1 *Golden Rule: two-body decay*

Starting from the simplied version of the Golden Rule for decays (introduced in the lecture),

$$\Gamma = \frac{S}{2m_1} \int |\mathfrak{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - \ldots - p_n) \prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3} \quad (1)$$

$$\text{with} \qquad p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2} \quad (2)$$

show that for the special case of a two-body decay $A \to B + C$, this expression can be simplified to

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2} |\mathfrak{M}|^2 \qquad \text{with} \qquad |\mathbf{p}| = |\mathbf{p}_2| = |\mathbf{p_3}| = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2)} \quad (3)$$

Symbols with indices 1, 2, 3 describe properties of particle A, B, C, in that order.

Hints:

For the integral(s) over $\mathbf{p}_3$, rewrite and simplify the delta function.

The integral(s) over $\mathbf{p}_2$ can be solved in spherical coordinates.

The matrix element $|\mathfrak{M}|$ depends only on the magnitude of a momentum, not on its direction (why?).

For the final integral over the momentum magnitude, use the transformation $u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2}$

You can use

$$m_1 = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2} \to r = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2)} \quad (4)$$

Exercise 2 *Toy model: two-body decay*

Consider the decay $A \to B + C$ in the toy model introduced in the lecture. The model contains three spinless particles A, B and C and a vertex between A, B and C with the coupling g.

Derive

(i) the matrix element $\mathfrak{M}$,

(ii) the total decay width Γ and

(iii) the differential width $\mathrm{d}\Gamma/\mathrm{d}\Omega$

for this decay as a function of the coupling g and the moduli of the momenta $|\vec{p}_f|$ of each of the two decay products in the rest frame of A. From the lecture, use the Feynman rules for (i) and Fermi's Golden Rule for (ii) and (iii).

Exercise 3 *Toy model: Feynman diagrams for $A + A \to B + B$ scattering*

For the same toy model as in the previous exercise, for the scattering process $A + A \to B + B$, draw

(i) the Feynman diagrams at lowest order (here, with two vertices each). (Hint: There are two.)

(ii) the Feynman diagrams at next-to-lowest order (here, with four vertices each). It is sufficient to draw those without twisted external lines. (Hint: In total, there are 8 diagrams without twisted external lines, but only three distinct types.)

Exercise 4 MANDELSTAM *variables*

Consider the $2 \to 2$ scattering process $1 + 2 \to 3 + 4$.

(i) For the case of identical mass ($m_1 = m_2 = m_3 = m_4 \equiv m$) in the center-of-mass system, show that

$$s = 4\left(p^2 + m^2\right), \qquad t = -2p^2\left(1 - \cos\theta\right), \qquad u = -2p^2\left(1 + \cos\theta\right),$$

where $p = |\vec{p}_i| = |\vec{p}_f|$ is the modulus of the momentum of the incoming and outgoing particles, and $\theta = \angle\left(\vec{p}_1, \vec{p}_3\right)$ is the scattering angle.

(ii) For which scattering angle do t and u reach their minima / maxima?

Exercise 5 *Natural units I*

(i) Determine the value of the gravitational constant $G_N \approx 6.7 \times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$ in natural units of particle physics (eV or eV^{-2}).

(ii) Determine the PLANCK mass $M_P = \sqrt{1/G_N}$ in natural units.

(iii) Determine the PLANCK mass, Planck time, and Planck length in SI units.

Home exercises

Exercise 6 *Golden Rule: scattering* **7 Punkte**

Starting from the simplied version of the Golden Rule for scattering in the center-of-mass system (introduced in the lecture),

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 \cdot m_2)^2}} \int |\mathfrak{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \ldots - p_n) \prod_{j=3}^{n} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3} \quad (5)$$

$$\text{with} \qquad p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2} \quad (6)$$

show that for the special case of a $2 \to 2$ process $A + B \to C + D$, the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{S|\mathfrak{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p_f}|}{|\mathbf{p_i}|} \quad (7)$$

where $\mathbf{p_f}$ is the magnitude of either outgoing momentum, $\mathbf{p_i}$ the magnitude of either ingoing momentum, and $E_1 = p_1^0$ and $E_2 = p_2^0$ the energies of particles A and B. The indices 1, 2, 3, 4 correspond to particles A, B, C, D, in that order.

Hints:

You can start by using $\sqrt{(p_1 \cdot p_2)^2 - (m_1 \cdot m_2)^2} = (E_1 + E_2)|\mathbf{p}_1|$ (valid in the center-of-mass system).
For the integral(s) over $\mathbf{p}_4$, rewrite and simplify the delta function.
This time, the matrix element $|\mathfrak{M}|$ in general depends on the angle between two of the momenta. You can anyway use spherical coordinates to solve the integral over the magnitude of $\mathbf{p}_3$ as you do not need to solve the angular part of that integral.
You will also need equation (4) to solve that final integral.

Exercise 7 *Toy model: scattering* **5 Punkte**

Consider the toy model from the lecture and the in-class exercises.

(i) Using the Feynman rules, derive the matrix element $\mathfrak{M}_{\text{ges}}$ for the process $A + B \to A + B$ as a function of the Mandelstam variables, particle masses, and couplings. Hint: There are contributions from two diagrams.

(ii) Compute the associated differential cross section $\frac{d\sigma}{d\Omega}$ in the center-of-mass system assuming the two particles A and B to be of equal mass, and a massless particle C in the relativistic limit, i.e., $s, |u| \gg m_A^2,\ m_B^2$. Use Fermi's Golden Rule.
Express your result as function of the scattering angle $\theta = \angle(\vec{p}_1, \vec{p}_3)$ and the energy E of particle A.

Exercise 8 *Toy model: Feynman diagrams for $A + A \to A + A$ scattering* **3 Punkte**

For the same toy model, for the scattering process $A + A \to A + A$, draw all the lowest-order diagrams. (Hint: There are six diagrams, all of them with four vertices.)

Exercise 9 *Natural units II* **2 Punkte**

Determine the following quantities in natural units of particle physics (eV or eV^{-1}):

(i) The length and width of a regular football field ($100\,\text{m} \times 70\,\text{m}$),

(ii) The potential energy of a football ($400\,\text{g}$) at a height above the ground corresponding to the height of a football goal (about $2.5\,\text{m}$). Use $g = 10\,\text{m/s}^2$.

Exercise 10 *Relativistic kinematics for the two-body decay* **3 Punkte**

Consider the decay $A \to BC$ in the rest frame of particle A.

(i) Show that the following equation describes the energies of the outgoing particles as a function of the masses involved:

$$E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} \qquad E_C = \frac{m_A^2 - m_B^2 + m_C^2}{2m_A}$$

(ii) Using this result, show that the momentum of the outgoing particles $|\vec{p}_f| = |\vec{p}_B| = |\vec{p}_C|$ is given by:

$$|\vec{p}_f| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2} \quad .$$