Supersymmetry

Theoretical Introduction

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Supersymmetry Transformation

A SUSY transformation turns a boson state into a fermion state, and vice versa.

So the operator Q that generates such transformations acts, schematically, like:

$$Q\left|\text{Boson}\right\rangle = \left|\text{Fermion}\right\rangle; \qquad Q\left|\text{Fermion}\right\rangle = \left|\text{Boson}\right\rangle$$

This means that Q must be an anticommuting spinor. This is an intrinsically complex object, so $Q^\dagger$ is also a distinct symmetry generator:

$$Q^\dagger\left|\text{Boson}\right\rangle = \left|\text{Fermion}\right\rangle; \qquad Q^\dagger\left|\text{Fermion}\right\rangle = \left|\text{Boson}\right\rangle$$

The possible forms for such theories are highly restricted by the Haag-Lopuszanski-Sohnius extension of the Coleman-Mandula Theorem.

In a 4-dimensional theory with chiral fermions (like the Standard Model) and non-trivial scattering, then Q carries spin-$1/2$ with L helicity, and $Q^\dagger$ has spin-$1/2$ with R helicity, and they must satisfy...

The Supersymmetry Algebra

$$
\begin{aligned}
\{Q, Q^\dagger\} &= P^\mu \\
\{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0 \\
[P^\mu, Q] &= [P^\mu, Q^\dagger] = 0 \\
[T^a, Q] &= [T^a, Q^\dagger] = 0
\end{aligned}
$$

Here $P^\mu = (H, \vec{\mathbf{p}})$ is the generator of spacetime translations, and T^a are the gauge generators. (This is schematic, with spinor indices suppressed for now. We will restore them later.)

The single-particle states of the theory fall into irreducible representations of this algebra, called **supermultiplets**. Fermion and boson members of a given supermultiplet are **superpartners** of each other. By definition, if $|\Omega\rangle$ and $|\Omega'\rangle$ are superpartners, then $|\Omega'\rangle$ is equal to some combination of $Q, Q^\dagger$ acting on $|\Omega\rangle$.

Therefore, since P^2 and T^a commute with $Q, Q^\dagger$, all members of a given supermultiplet must have the same (mass)^2 and gauge quantum numbers.

Number of Fermions and Bosons in Supermultiplet

Each supermultiplet contains equal numbers of fermions and bosons

Proof: Consider the operator $(-1)^{2S}$ where S is spin angular momentum. Then

$$(-1)^{2S} = \begin{cases} -1 & \text{acting on fermions} \\ +1 & \text{acting on bosons} \end{cases}$$

So, $(-1)^{2S}$ must anticommute with Q and $Q^\dagger$. Now consider all states $|i\rangle$ in a given supermultiplet with the same momentum eigenvalue $p^\mu \neq 0$. These form a complete set of states, so $\sum_j |j\rangle\langle j| = 1$. Now do a little calculation:

$$\begin{aligned}
p^\mu \mathrm{Tr}[(-1)^{2S}] = \sum_i \langle i|(-1)^{2S} P^\mu |i\rangle &= \sum_i \langle i|(-1)^{2S} QQ^\dagger |i\rangle + \sum_i \langle i|(-1)^{2S} Q^\dagger Q|i\rangle \\
&= \sum_i \langle i|(-1)^{2S} QQ^\dagger |i\rangle + \sum_i \sum_j \langle i|(-1)^{2s} Q^\dagger |j\rangle\langle j|Q|i\rangle \\
&= \sum_i \langle i|(-1)^{2S} QQ^\dagger |i\rangle + \sum_j \langle j|Q(-1)^{2S} Q^\dagger |j\rangle \\
&= \sum_i \langle i|(-1)^{2S} QQ^\dagger |i\rangle - \sum_j \langle j|(-1)^{2S} QQ^\dagger |j\rangle \\
&= 0.
\end{aligned}$$

The trace just counts the number of boson minus the number of fermion degrees of freedom in the supermultiplet. Therefore, $p^\mu(n_B - n_F) = 0$.

Types of Supermultiplets

Chiral (or “Scalar” or “Matter” or “Wess-Zumino”) supermultiplet:

1 two-component Weyl fermion, helicity $\pm\frac{1}{2}$. $(n_F = 2)$

2 real spin-0 scalars = 1 complex scalar. $(n_B = 2)$

The Standard Model quarks, leptons and Higgs bosons must fit into these.

Gauge (or “Vector”) supermultiplet:

1 two-component Weyl fermion gaugino, helicity $\pm\frac{1}{2}$. $(n_F = 2)$

1 real spin-1 massless gauge vector boson. $(n_B = 2)$

The Standard Model $\gamma, Z, W^\pm, g$ must fit into these.

Gravitational supermultiplet:

1 two-component Weyl fermion gravitino, helicity $\pm\frac{3}{2}$. $(n_F = 2)$

1 real spin-2 massless graviton. $(n_B = 2)$

Each quark or charged lepton is 1 Dirac = 2 Weyl fermions

Electron: $\Psi_e = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$ $\begin{matrix} \leftarrow \text{two-component Weyl LH fermion} \\ \leftarrow \text{two-component Weyl RH fermion} \end{matrix}$

Each of e_L and e_R is part of a chiral supermultiplet, so each has a complex, spin-0 superpartner, called $\widetilde{e}_L$ and $\widetilde{e}_R$ respectively. They are called the "left-handed selectron" and "right-handed selectron", although they carry no spin.

The conjugate of a right-handed Weyl spinor is a left-handed Weyl spinor. Define two-component left-handed Weyl fields: $e \equiv e_L$ and $\bar{e} \equiv e_R^\dagger$. So, there are two left-handed chiral supermultiplets for the electron:

$$(e,\ \widetilde{e}_L) \quad \text{and} \quad (\bar{e},\ \widetilde{e}_R^*).$$

The other charged leptons and quarks are similar. We do not need ν_R in the Standard Model, so there is only one neutrino chiral supermultiplet for each family:

$$(\nu_e,\ \widetilde{\nu}_e).$$

Minimal Supersymmetric Standard Model (MSSM)

- **The MSSM contains the minimum number of particles and the minimum number of couplings in the superpotential needed to obtain realistic, phenomenologically reasonable results.**

Chiral supermultiplets

	Spin 0	Spin 1/2	$SU(3)_C, SU(2)_L, U(1)_Y$
Q	$(\tilde{u}_L, \tilde{d}_L)$	(u_L, d_L)	$(3, 2, \frac{1}{6})$
$\bar{u}$	$\tilde{u}_R^*$	$u_R^\dagger$	$(\bar{3}, 1, -\frac{2}{3})$
$\bar{d}$	$\tilde{d}_R^*$	$d_R^\dagger$	$(\bar{3}, 1, \frac{1}{3})$
L	$(\tilde{\nu}, \tilde{e}_L)$	(ν, e_L)	$(1, 2, -\frac{1}{2})$
$\bar{e}$	$\tilde{e}_R^*$	$e_R^\dagger$	(1,1,1)
H_u	(H_u^+, H_u^0)	$(\tilde{H}_u^+, \tilde{H}_u^0)$	$(1, 2, +\frac{1}{2})$
H_d	(H_d^0, H_d^-)	$(\tilde{H}_d^0, \tilde{H}_d^-)$	$(1, 2, -\frac{1}{2})$

Gauge supermultiplets

	Spin 1/2	Spin 1	$SU(3)_C, SU2_L, U(1)_Y$
Gluino, Gluon	$\tilde{g}$	g	(8,1,0)
Wino, W-Boson	$\tilde{W}^\pm, \tilde{W}^0$	$W^\pm, W^0$	(1,3,0)
Bino, B-Boson	$\tilde{B}^0$	B^0	(1,1,0)

Simplest SUSY Model: Free Chiral Supermultiplet

The minimum particle content for a SUSY theory is a complex scalar ϕ and its superpartner fermion ψ. We must at least have kinetic terms for each, so:

$$S = \int d^4x \, (\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}})$$

$$\mathcal{L}_{\text{scalar}} = -\partial^\mu \phi^* \partial_\mu \phi \qquad \mathcal{L}_{\text{fermion}} = -i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi$$

A SUSY transformation should turn ϕ into ψ, so try:

$$\delta\phi = \epsilon\psi; \qquad \delta\phi^* = \epsilon^\dagger \psi^\dagger$$

where $\epsilon =$ infinitesimal, anticommuting, constant spinor, with dimension $[\text{mass}]^{-1/2}$, that parameterizes the SUSY transformation. Then we find:

$$\delta\mathcal{L}_{\text{scalar}} = -\epsilon\partial^\mu\psi\partial_\mu\phi^* - \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi.$$

We would like for this to be canceled by an appropriate SUSY transformation of the fermion field. . .

Free Wess-Zumino Model

To have any chance, $\delta\psi$ should be linear in $\epsilon^\dagger$ and in ϕ, and must contain one spacetime derivative. There is only one possibility, up to a multiplicative constant:

$$\delta\psi_\alpha = i(\sigma^\mu \epsilon^\dagger)_\alpha \partial_\mu \phi; \qquad \delta\psi^\dagger_{\dot\alpha} = -i(\epsilon\sigma^\mu)_{\dot\alpha}\partial_\mu \phi^*$$

With this guess, one obtains:

$$\delta\mathcal{L}_{\rm fermion} = -\delta\mathcal{L}_{\rm scalar} + (\text{total derivative})$$

so the action S is indeed invariant under the SUSY transformation, justifying the guess of the multiplicative factor. This is called the free Wess-Zumino model.

Furthermore, if we take the commutator of two SUSY transformations:

$$\delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) = i(\epsilon_1\sigma^\mu\epsilon_2 - \epsilon_2\sigma^\mu\epsilon_1)\partial_\mu\phi$$

Since ∂_μ corresponds to the spacetime 4-momentum P_μ, this has exactly the form demanded by the SUSY algebra discussed earlier.

The fact that two SUSY transformations give back another symmetry (namely a spacetime translation) means that the SUSY algebra "closes".

If we do the same check for the fermion ψ:

$$\begin{aligned}\delta_{\epsilon_2}(\delta_{\epsilon_1}\psi_\alpha) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\psi_\alpha) &= i(\epsilon_1\sigma^\mu\epsilon_2 - \epsilon_2\sigma^\mu\epsilon_1)\partial_\mu\psi_\alpha \\ &\quad -i\epsilon_{1\alpha}(\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi) + i\epsilon_{2\alpha}(\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi)\end{aligned}$$

The first line is expected, but the second line only vanishes on-shell (when the classical equations of motion are satisfied). This seems like a problem, since we want SUSY to be a valid symmetry of the quantum theory (off-shell)!

To show that there is no problem, we introduce another bosonic spin-0 field, F, called an auxiliary field. Its Lagrangian density is:

$$\mathcal{L}_{\text{aux}} = F^* F$$

Note that F has no kinetic term, and has dimensions [mass]^2, unlike an ordinary scalar field. It has the not-very-exciting equations of motion $F = F^* = 0$.

Auxiliary Field F

The auxiliary field F does not affect the dynamics, classically or in the quantum theory. But it does appear in modified SUSY transformation laws:

$$\begin{aligned} \delta\phi &= \epsilon\psi \\ \delta\psi_\alpha &= i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu\phi + \epsilon_\alpha F \\ \delta F &= i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi \end{aligned}$$

Now the total Lagrangian

$$\mathcal{L} = -\partial^\mu\phi^*\partial_\mu\phi - i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + F^*F$$

is still invariant, and also one can now check:

$$\delta_{\epsilon_2}(\delta_{\epsilon_1}X) - \delta_{\epsilon_1}(\delta_{\epsilon_2}X) \;=\; i(\epsilon_1\sigma^\mu\epsilon_2 - \epsilon_2\sigma^\mu\epsilon_1)\partial_\mu X$$

for each of $X = \phi, \phi^*, \psi, \psi^\dagger, F, F^*$, without using equations of motion.

So in the “modified” theory, SUSY does close off-shell as well as on-shell.

Lagrangian for Chiral Supermultiplets

Lagrangian describing a collection of free, massless chiral supermultiplets

$$\mathcal{L} = -\partial^\mu \phi^{*i} \partial_\mu \phi_i - i\psi^{\dagger i} \overline{\sigma}^\mu \partial_\mu \psi_i + F^{*i} F_i$$

is invariant under the transformations parameterized by a constant spinor ϵ_α:

$$\begin{aligned} \delta\phi_i &= \epsilon\psi_i, \\ \delta(\psi_i)_\alpha &= -i(\sigma^\mu \epsilon^\dagger)_\alpha \partial_\mu \phi_i + \epsilon_\alpha F_i \\ \delta F_i &= -i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi_i \end{aligned}$$

Now we try to add to this a Lagrangian describing interactions:

$$\mathcal{L}_{\text{int}} = \left(-\tfrac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + x^{ij} F_i F_j\right) + \text{c.c.} + U$$

where, to be renormalizable, W^{ij}, W^i, x^{ij}, and U are polynomials in ϕ_i, ϕ^{*i} with degrees 1, 2, 0, and 4, respectively.

The Superpotential

Now one can compute $\delta\mathcal{L}$ under the SUSY transformation, and require that it be a total derivative, so that the action $S = \int d^4x\mathcal{L}$ is invariant.

This turns out to work if and only if $x^{ij} = 0$ and $U = 0$, and:

$$W^{ij} = \frac{\partial^2 W}{\partial\phi_i\partial\phi_j} = M^{ij} + y^{ijk}\phi_k$$
$$W^{i} = \frac{\partial W}{\partial\phi_i} = M^{ij}\phi_j + \tfrac{1}{2}y^{ijk}\phi_j\phi_k$$

where we have defined a useful function:

$$W = \tfrac{1}{2}M^{ij}\phi_i\phi_j + \tfrac{1}{6}y^{ijk}\phi_i\phi_j\phi_k$$

called the **superpotential**. Note that it does not depend on ϕ^{*i}, only the ϕ_i. It is an analytic function of the scalar fields treated as complex variables.

The superpotential W contains masses M^{ij} and couplings y^{ijk}, which are each automatically symmetric under interchange of i, j, k.

Supersymmetry is very restrictive; you cannot just do anything you want!

Eliminating the Auxiliary Field *F*

The Lagrangian terms involving auxiliary fields F_i, F^{*i} are:

$$\mathcal{L} = F^{*i}F_i + W^iF_i + W_i^*F^{*i}$$

So the equations of motion are now:

$$F^{*i} \;=\; -W^i = M^{ij}\phi_j + \tfrac{1}{2}y^{ijk}\phi_j\phi_k,$$

This is still algebraic; no spacetime derivatives. By eliminating the auxiliary fields, we get the complete Lagrangian:

$$\begin{aligned}\mathcal{L} \;=\;& -\partial^\mu\phi^{*i}\partial_\mu\phi_i - V(\phi_i,\phi^{*i})\\ & -i\psi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\psi_i - \tfrac{1}{2}\left(M^{ij}\psi_i\psi_j + y^{ijk}\phi_i\psi_j\psi_k + \text{c.c.}\right)\end{aligned}$$

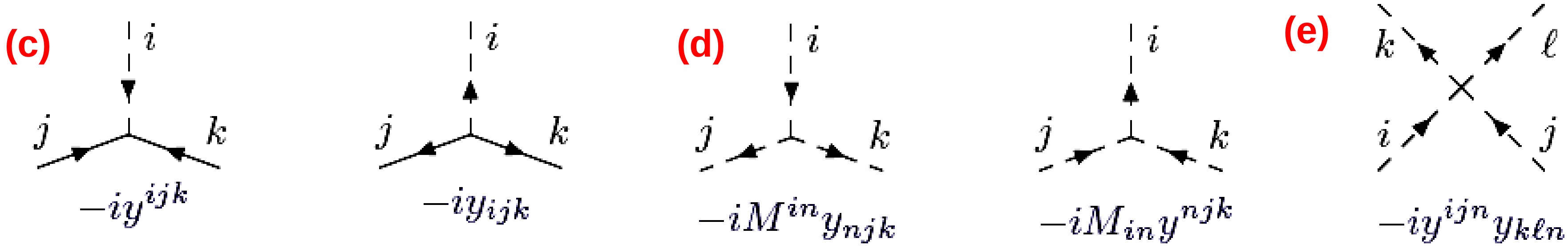
where the scalar potential is:

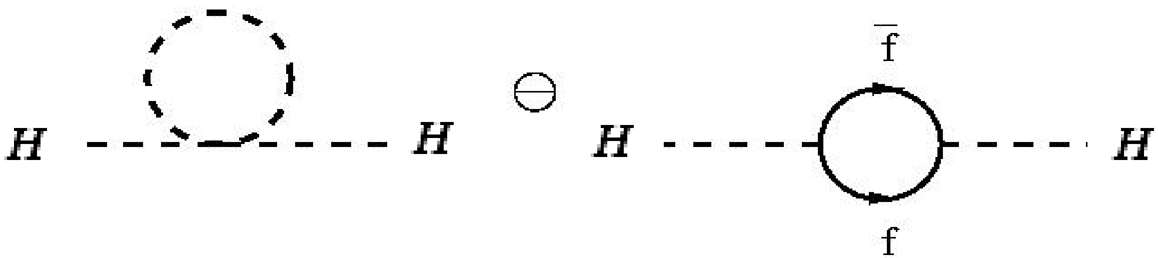
$$\begin{aligned}V(\phi_i,\phi^{*i}) = F_iF^{*i} = W^iW_i^* = M_{ik}M^{kj}\phi^{*i}\phi_j + \frac{1}{2}M^{in}y_{jkn}\phi_i\phi^{*j}\phi^{*k}\\ +\frac{1}{2}M_{in}y^{jkn}\phi^{*i}\phi_j\phi_k + \frac{1}{4}y^{ijn}y_{kln}\phi_i\phi_j\phi^{*k}\phi^{*l}\end{aligned}$$

$$\mathcal{L} = -\partial^\mu \phi^{*i} \partial_\mu \phi_i - V(\phi_i, \phi^{*i})$$
$$-i\psi^{\dagger i}\overline{\sigma}^\mu \partial_\mu \psi_i - \tfrac{1}{2}\big(\underbrace{M^{ij}\psi_i\psi_j}_{\text{(a)}} + \underbrace{y^{ijk}\phi_i\psi_j\psi_k}_{\text{(c)}} + \text{c.c.}\big)$$

$$V(\phi_i, \phi^{*i}) = F_i F^{*i} = W^i W^*_i = \overbrace{M_{ik}M^{kj}\phi^{*i}\phi_j}^{\text{(b)}} + \overbrace{\frac{1}{2}M^{in}y_{jkn}\phi_i\phi^{*j}\phi^{*k}}^{\text{(d)}}$$
$$+\underbrace{\frac{1}{2}M_{in}y^{jkn}\phi^{*i}\phi_j\phi_k}_{\text{(d)}} + \underbrace{\frac{1}{4}y^{ijn}y_{kln}\phi_i\phi_j\phi^{*k}\phi^{*l}}_{\text{(e)}}$$

Mass terms:





The systematic cancellation of loop corrections to the Higgs mass squared requires the type of conspiracy that is better known to physicists as a **symmetry**.

Fermion loops and boson loops gave contributions with opposite signs:

$$\Delta m_H^2 = -\frac{\lambda_f^2}{16\pi^2}(2M_{\mathrm{UV}}^2) + \ldots \qquad \text{(Dirac fermion)}$$

$$\Delta m_H^2 = +\frac{\lambda_S}{16\pi^2}M_{\mathrm{UV}}^2 + \ldots \qquad \text{(complex scalar)}$$

So we need a **SUPERSYMMETRY** = a symmetry between fermions and bosons.

It turns out that this makes the cancellation not only possible, but automatic.

Excursion: SUSY and the Hierarchy Problem

For a clue as to the nature of SUSY breaking, return to our motivation in the Hierarchy Problem. The Higgs mass parameter gets corrections from each chiral supermultiplet:

$$\Delta m_H^2 = \frac{1}{16\pi^2}(\lambda_S - \lambda_F^2)M_{\rm UV}^2 + \ldots$$

The corresponding formula for Higgsinos has no term proportional to $M_{\rm UV}^2$; fermion masses always diverge at worst like $\ln(M_{\rm UV})$. Therefore, if supersymmetry were exact and unbroken, it must be that:

$$\lambda_S = \lambda_F^2,$$

in other words, the dimensionless $(\text{scalar})^4$ couplings are the squares of the (scalar)-(fermion)-(antifermion) couplings.

If we want SUSY to be a solution to the hierarchy problem, we must demand that this is still true even after SUSY is broken:

The breaking of supersymmetry must be “soft”. This means that it does not change the dimensionless terms in the Lagrangian.

A gauge or vector supermultiplet contains a gauge boson A^a_μ and a gaugino λ^a_α.

$$\begin{pmatrix} A^a_\mu \\ \lambda^a \end{pmatrix}$$

← **gauge boson**

← **gaugino**

The index a runs over the gauge group generators $[1, 2, \ldots, 8$ for $SU(3)_C$; $1, 2, 3$ for $SU(2)_L$; 1 for $U(1)_Y]$.

Suppose the gauge coupling constant is g and the structure constants of the group are f^{abc}. The Lagrangian for the gauge supermultiplet is:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} - i\lambda^{\dagger a} \overline{\sigma}^\mu \nabla_\mu \lambda^a + \frac{1}{2} D^a D^a$$

where D^a is a real spin-0 auxiliary field with no kinetic term, and

$$\nabla_\mu \lambda^a = (\partial_\mu \lambda^a - g f^{abc} A^b_\mu \lambda^c)$$

The action is invariant under the SUSY transformation:

$$\delta A^a_\mu = -\frac{1}{\sqrt{2}} \left(\epsilon^\dagger \overline{\sigma}_\mu \lambda^a + \lambda^{\dagger a} \overline{\sigma}_\mu \epsilon \right),$$

$$\delta \lambda^a_\alpha = -\frac{i}{2\sqrt{2}} (\sigma^\mu \overline{\sigma}^\nu \epsilon)_\alpha \, F^a_{\mu\nu} + \frac{1}{\sqrt{2}} \epsilon_\alpha \, D^a,$$

$$\delta D^a = \frac{i}{\sqrt{2}} \left(\epsilon^\dagger \overline{\sigma}^\mu \nabla_\mu \lambda^a - \nabla_\mu \lambda^{\dagger a} \overline{\sigma}^\mu \epsilon \right).$$

Combined Chiral and Gauge Lagrangian

To make a gauge-invariant supersymmetric Lagrangian involving both gauge and chiral supermultiplets, one must turn the ordinary derivatives into covariant ones:

$$\begin{aligned} \partial_\mu \phi_i &\rightarrow \nabla_\mu \phi_i = \partial_\mu \phi_i + igA_\mu^a (T^a \phi)_i \\ \partial_\mu \psi_i &\rightarrow \nabla_\mu \psi_i = \partial_\mu \psi_i + igA_\mu^a (T^a \psi)_i \end{aligned}$$

One must also add three new terms to the Lagrangian:

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{chiral}} - \sqrt{2} g (\phi^* T^a \psi) \lambda^a - \sqrt{2} g \lambda^{\dagger a} (\psi^\dagger T^a \phi) \\ & + g (\phi^* T^a \phi) D^a . \end{aligned}$$

You can check (after some algebra) that this full Lagrangian is now invariant under both SUSY transformations and gauge transformations.

Eliminating the Auxiliary Field D

The part of the Lagrangian involving the auxiliary fields D^a is:

$$\mathcal{L} = \tfrac{1}{2} D^a D^a + g(\phi^* T^a \phi)$$

So the D^a obey purely algebraic equations of motion $D^a = -g(\phi^* T^a \phi)$, and so can be eliminated from the theory. The resulting scalar potential is:

$$\begin{aligned} V(\phi_i, \phi^{*i}) &= F^{*i} F_i + \tfrac{1}{2} D^a D^a \\ &= \underbrace{W_i^* W^i}_{\text{F-term}} + \underbrace{\tfrac{1}{2} \sum_a g_a^2 (\phi^* T^a \phi)^2}_{\text{D-term}} \end{aligned}$$

- Since V is a sum of squares, it is automatically ≥ 0.
- The scalar potential in SUSY theories is completely determined by the fermion masses, Yukawa couplings, and gauge couplings.

But both of these statements will be modified when we break SUSY.

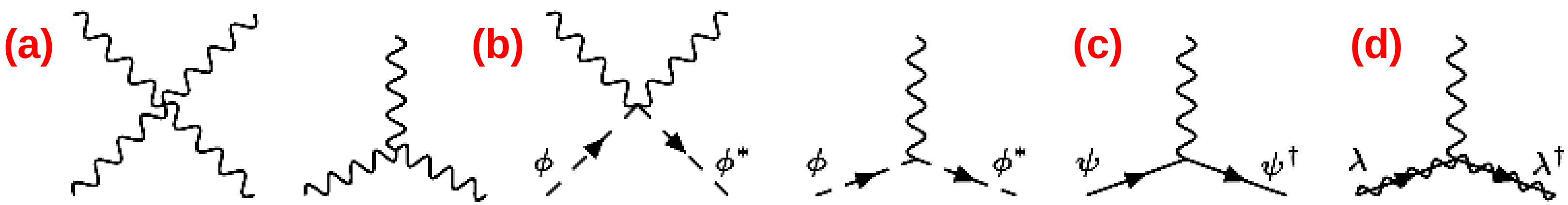
Gauge Interactions

$$\mathcal{L} = \overbrace{-\tfrac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu}}^{(a)} \overbrace{- i\lambda^{\dagger a}\overline{\sigma}^\mu \nabla_\mu \lambda^a}^{(d)} \overbrace{- \sqrt{2}g(\phi^* T^a \psi)\lambda^a - \sqrt{2}g\lambda^{\dagger a}(\psi^\dagger T^a \phi)}^{(e)}$$

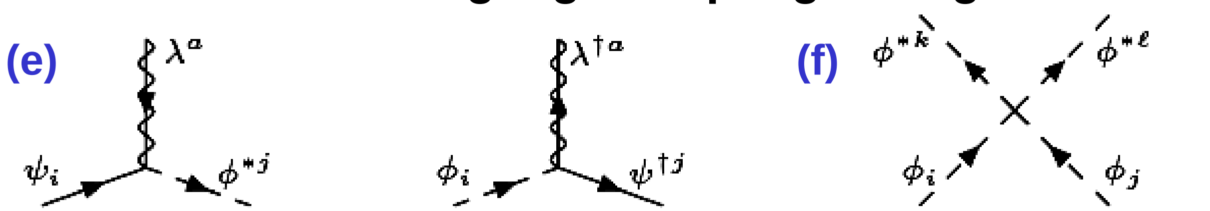
$$\underbrace{-\nabla_\mu \phi^{*i}\nabla_\mu \phi_i}_{(b)} \underbrace{- i\psi^{\dagger i}\overline{\sigma}^\mu \nabla_\mu \psi_i}_{(c)} - V(\phi_i, \phi^{*i})$$

$$V(\phi_i, \phi^{*i}) \;=\; W_i^* W^i + \underbrace{\tfrac{1}{2}\sum_a g_a^2 (\phi^* T^a \phi)^2}_{(f)}$$

- **Interactions resulting from ordinary gauge invariance:**



- **Additional interactions with gauge coupling strength:**



$-i\sqrt{2}g_a(T^a)_i{}^j$ $\quad$ $-i\sqrt{2}g_a(T^a)_i{}^j$ $\quad$ $-ig_a^2(T_i^{ak}T_j^{a\ell}+T_i^{a\ell}T_j^{ak})$

Summary: SUSY Lagrangian

$$\mathcal{L}_{SUSY} = -\underbrace{\mathrm{i}\bar{\Psi}\bar{\sigma}^{\mu}D_{\mu}\Psi}_{\text{Fermionen}}\underbrace{-D^{\mu}\Phi^{*}D_{\mu}\Phi}_{\text{Skalare}}\quad\underbrace{-\frac{1}{2}(W^{ij}\Psi_i\Psi_j + W^{ij*}\Psi^{it}\Psi^{jt})}_{\text{Yukawa Kopplung und Fermionmasseterme}}$$

$$\underbrace{-\frac{1}{2}W^{i}W^{*}_{i} + \frac{1}{2}g_a^2(\Phi^{*}T^{a}\Phi)^2}_{\text{skalares Potential}}\underbrace{-\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu a}}_{\text{Eichfelder}}\underbrace{-\mathrm{i}\lambda^{ta}\bar{\sigma}^{\mu}D_{\mu}\lambda^{a}}_{\text{Gauginos}}$$

$$\underbrace{-\sqrt{2}g((\Phi^{*}T^{a}\Psi)\lambda^{a} + \lambda^{at}(\Psi^{t}T^{a}\Phi))}_{\text{zusätzliche Kopplungen}} + \mathcal{L}_{soft}$$

Masses of Sparticles

Recall that if supersymmetry were an exact symmetry, then superpartners would have to be exactly degenerate with each other. For example,

$$m_{\tilde{e}_L} = m_{\tilde{e}_R} = m_e = 0.511 \text{ GeV}$$

$$m_{\tilde{u}_L} = m_{\tilde{u}_R} = m_u$$

$$m_{\tilde{g}} = m_{\text{gluon}} = 0 + \text{QCD-scale effects}$$

etc.

But new particles with these properties have been ruled out long ago, so:

Supersymmetry must be broken in the vacuum state chosen by Nature.

Supersymmetry is thought to be spontaneously broken and therefore hidden, the same way that the electroweak symmetry is hidden from very low-energy experiments.

SUSY Breaking and Hierarchy Problem

The effective Lagrangian of the MSSM can therefore be written in the form:

$$\mathcal{L} = \mathcal{L}_{\rm SUSY} + \mathcal{L}_{\rm soft}$$

- $\mathcal{L}_{\rm SUSY}$ contains all of the gauge interactions and Yukawa interactions dimensionless scalar couplings, and preserves exact supersymmetry
- $\mathcal{L}_{\rm soft}$ violates supersymmetry, and contains only mass terms and couplings with *positive* mass dimension.

If $m_{\rm soft}$ is the largest mass scale in $\mathcal{L}_{\rm soft}$, then by dimensional analysis,

$$\Delta m_H^2 = m_{\rm soft}^2 \left[\frac{\lambda}{16\pi^2} \ln(M_{\rm UV}/m_{\rm soft}) + \ldots \right],$$

where λ stands for dimensionless couplings. This is because Δm_H^2 must vanish in the limit $m_{\rm soft} \to 0$, in which SUSY is restored. Therefore, we expect that $m_{\rm soft}$ should not be much larger than roughly 1000 GeV.

SM vs. SUSY Masses

One might also ask if there is any good reason why the superpartners of the Standard Model particles should be heavy enough to have avoided discovery so far. There is!

- All of the particles in the MSSM that have been discovered as of 1995 (quarks, leptons, gauge bosons) would be exactly massless if the electroweak symmetry were not broken. So their masses are expected to be at most of order $v = 175$ GeV, the electroweak breaking scale. **In other words, they are required to be light.**

- All of the particles in the MSSM that have not yet been discovered (squarks, sleptons, gauginos, Higgsinos, Higgs scalars) can get a mass even without electroweak symmetry breaking. **They are not required to be light.**

Soft SUSY-Breaking Lagrangians

It has been shown rigorously that the quadratic sensitivity to $M_{\rm UV}$ does not arise in SUSY theories with these terms added in:

$$\mathcal{L}_{\rm soft} = -\frac{1}{2}\left(M_a\,\lambda^a\lambda^a + \text{c.c.}\right) - (m^2)^i_j\phi^{*j}\phi_i - \left(\frac{1}{2}b^{ij}\phi_i\phi_j + \frac{1}{6}a^{ijk}\phi_i\phi_j\phi_k + \text{c.c.}\right),$$

They consist of:

- gaugino masses M_a,
- scalar (mass)^2 terms $(m^2)^j_i$ and b^{ij},
- (scalar)^3 couplings a^{ijk}

Building a Realistic SUSY Model

- Choose a gauge symmetry group.
  (In the MSSM, this is already done: $SU(3)_C \times SU(2)_L \times U(1)_Y$.)
- Choose a superpotential W; must be invariant under the gauge symmetry.
  (In the MSSM, this is almost already done: Yukawa couplings are dictated by the observed fermion masses.)
- Choose a soft SUSY-breaking Lagrangian, or else choose a method for spontaneous SUSY breakdown.
  (This is where almost all of the arbitrariness in the MSSM is.)

Let's do this for the MSSM now, and then explore the consequences.

$$W = \frac{1}{2} M^{ij} \phi_i \phi_j + \frac{1}{6} y^{ijk} \phi_i \phi_j \phi_k$$

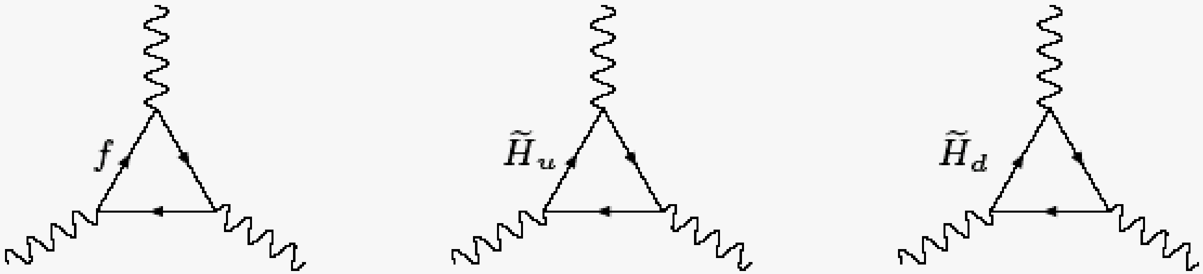
$$W_{\text{MSSM}} = \tilde{\bar{u}} \mathbf{y_u} \tilde{Q} H_u - \tilde{\bar{d}} \mathbf{y_d} \tilde{Q} H_d - \tilde{\bar{e}} \mathbf{y_e} \tilde{L} H_d + \mu H_u H_d$$

The objects H_u, H_d, $\tilde{Q}$, $\tilde{L}$, $\tilde{\bar{u}}$, $\tilde{\bar{d}}$, $\tilde{\bar{e}}$ appearing here are the scalar fields appearing in the left-handed chiral supermultiplets. Recall that $\bar{u}$, $\bar{d}$, $\bar{e}$ are the conjugates of the right-handed parts of the quark and lepton fields.

The dimensionless Yukawa couplings $\mathbf{y_u}$, $\mathbf{y_d}$ and $\mathbf{y_e}$ are 3×3 matrices in family space. Up to a normalization, and higher-order quantum corrections, they are the same as in the Standard Model. (All gauge and family indices are suppressed.)

$\tilde{\bar{u}} \mathbf{y_u} \tilde{Q} H_d^*$ and $\tilde{\bar{d}} \mathbf{y_d} \tilde{Q} H_u^*$ are not allowed in the superpotential, since they are not analytic.

1) Anomaly Cancellation



$$Tr\left[Y^3\right] = 0$$

$$\sum_{\text{SM fermions}} Y_f^3 = 0 \qquad + 2\left(\frac{1}{2}\right)^3 \qquad + 2\left(-\frac{1}{2}\right)^3 = 0$$

This anomaly cancellation occurs if and only if **both** $\widetilde{H}_u$ and $\widetilde{H}_d$ higgsinos are present. Otherwise, the electroweak gauge symmetry would not be allowed!

2) Quark and Lepton masses

Only the H_u Higgs scalar can give masses to charge $+2/3$ quarks (top).

Only the H_d Higgs scalar can give masses to charge $-1/3$ quarks (bottom) and the charged leptons.

Note that, as promised earlier, we need both H_u and H_d, because terms like $\bar{\tilde{u}}\mathbf{y_u}\tilde{Q}H_d^*$ and $\bar{\tilde{d}}\mathbf{y_d}\tilde{Q}H_u^*$ are not allowed in the superpotential, since they are not analytic.

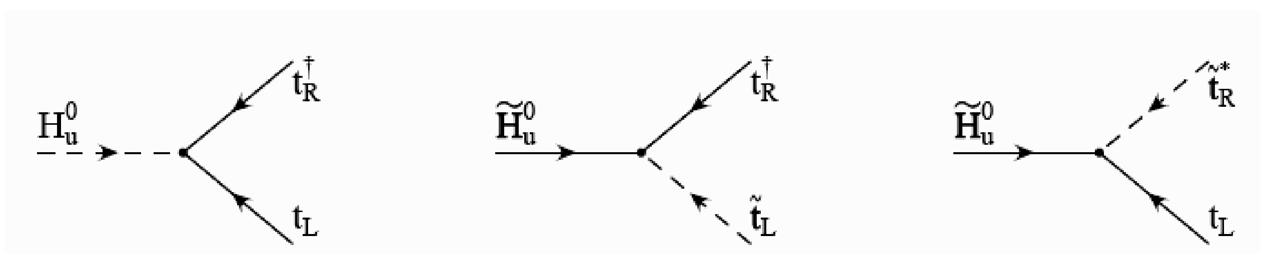
MSSM Superpotential

In the approximation that only the t, b, τ Yukawa couplings are included:

$$\mathbf{y_u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix}; \qquad \mathbf{y_d} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_b \end{pmatrix}; \qquad \mathbf{y_e} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\tau \end{pmatrix}$$

the superpotential becomes

$$\begin{aligned} W_{\text{MSSM}} \approx\ & y_t(\bar{t}tH_u^0 - \bar{t}bH_u^+) - y_b(\bar{b}tH_d^- - \bar{b}bH_d^0) \\ & - y_\tau(\bar{\tau}\nu_\tau H_d^- - \bar{\tau}\tau H_d^0) + \mu(H_u^+H_d^- - H_u^0H_d^0) \end{aligned}$$



Note that the minus signs are arranged so that if the neutral Higgs scalars get positive VEVs $\langle H_u^0 \rangle = v_u$ and $\langle H_d^0 \rangle = v_d$, and the Yukawa couplings are defined positive, then the fermion masses are also positive:

$$m_t = y_t v_u; \qquad m_b = y_b v_d; \qquad m_\tau = y_\tau v_d.$$

Soft SUSY-Breaking Lagrangian in the MSSM

$$\begin{aligned}\mathcal{L}_{\rm soft}^{\rm MSSM} = & -\tfrac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B}\right) + \text{c.c.} \\ & -\left(\widetilde{\overline{u}}\,\mathbf{a_u}\,\widetilde{Q}H_u - \widetilde{\overline{d}}\,\mathbf{a_d}\,\widetilde{Q}H_d - \widetilde{\overline{e}}\,\mathbf{a_e}\,\widetilde{L}H_d\right) + \text{c.c.} \\ & -\widetilde{Q}^\dagger\,\mathbf{m^2_Q}\,\widetilde{Q} - \widetilde{L}^\dagger\,\mathbf{m^2_L}\,\widetilde{L} - \widetilde{\overline{u}}\,\mathbf{m^2_{\overline{u}}}\,\widetilde{\overline{u}}^\dagger - \widetilde{\overline{d}}\,\mathbf{m^2_{\overline{d}}}\,\widetilde{\overline{d}}^\dagger - \widetilde{\overline{e}}\,\mathbf{m^2_{\overline{e}}}\,\widetilde{\overline{e}}^\dagger \\ & - m^2_{H_u}H_u^*H_u - m^2_{H_d}H_d^*H_d - (bH_uH_d + \text{c.c.})\,.\end{aligned}$$

The first line gives masses to the MSSM gauginos (gluino $\widetilde{g}$, winos $\widetilde{W}$, bino $\widetilde{B}$).

The second line consists of (scalar)^3 interactions.

The third line is (mass)^2 terms for the squarks and sleptons.

The last line is Higgs (mass)^2 terms.

If SUSY is to solve the Hierarchy Problem, we expect:

$$M_1,\, M_2,\, M_3,\, \mathbf{a_u},\, \mathbf{a_d},\, \mathbf{a_e} \sim m_{\rm soft};$$

$$\mathbf{m^2_{\widetilde{Q}}},\, \mathbf{m^2_{\widetilde{L}}},\, \mathbf{m^2_{\widetilde{\overline{u}}}},\, \mathbf{m^2_{\widetilde{\overline{d}}}},\, \mathbf{m^2_{\widetilde{\overline{e}}}},\, m^2_{H_u},\, m^2_{H_d},\, b \sim m^2_{\rm soft}$$

where $m_{\rm soft} \lesssim 1$ TeV.

A Comment on MSSM Parameters

The squark and slepton squared masses and (scalar)^3 couplings are 3×3 matrices in family space. The soft SUSY-breaking Lagrangian of the MSSM contains 105 new parameters not found in the Standard Model.

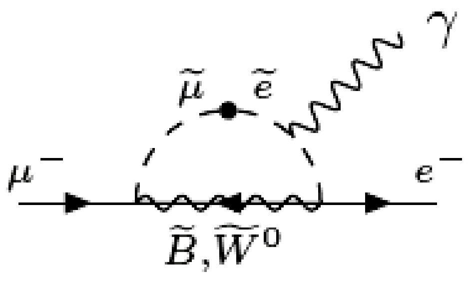
Most of what we do not already know about SUSY is expressed by the question: "How is supersymmetry broken?"

Many proposals exist. None are completely convincing.

The question can be answered experimentally by discovering the pattern of Higgs and squark and slepton and gaugino masses, because they are the main terms in the SUSY-breaking Lagrangian.

Fortunately, we already know that the MSSM soft terms cannot be arbitrary, because of experimental constraints on flavor violation.

For example, if there is a smuon-selectron mixing (mass)^2 term $\mathcal{L} = -m^2_{\tilde{\mu}_L \tilde{e}_L} \tilde{e}_L \tilde{\mu}^*_L$, and $\tilde{M} =$ Max$[m_{\tilde{e}_L}, m_{\tilde{e}_R}, M_2]$, then by calculating this one-loop diagram, one finds the decay width:



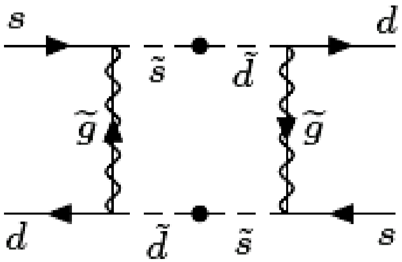
$$\Gamma(\mu^- \to e^- \gamma) = 5 \times 10^{-21}\ \text{MeV} \left(\frac{m^2_{\tilde{\mu}_L \tilde{e}_L}}{\tilde{M}^2} \right)^2 \left(\frac{100\ \text{GeV}}{\tilde{M}} \right)^4$$

For comparison, the experimental limit is (from MEGA at LAMPF):

$$\Gamma(\mu^- \to e^- \gamma) < 3.6 \times 10^{-27}\ \text{MeV}.$$

So the amount of smuon-selectron mixing in the soft Lagrangian is limited by:

$$\left(\frac{m^2_{\tilde{\mu}_L \tilde{e}_L}}{\tilde{M}^2} \right) < 10^{-3} \left(\frac{\tilde{M}}{100\ \text{GeV}} \right)^2$$



$K^0 \leftrightarrow \overline{K^0}$

Another example: $K^0 \leftrightarrow \overline{K^0}$ mixing.

Let $\mathcal{L} = -m^2_{\tilde{d}_L\tilde{s}_L}\tilde{d}_L\tilde{s}^*_L$ be the flavor-violating term, and $\tilde{M} = \text{Max}[m_{\tilde{d}_L}, m_{\tilde{s}_L}, m_{\tilde{g}}]$.

Comparing this diagram with Δm_{K^0} gives:

$$\frac{m^2_{\tilde{d}_L\tilde{s}_L}}{\tilde{M}^2} < 0.04\left(\frac{\tilde{M}}{500\text{ GeV}}\right)$$

The experimental values of ϵ and ϵ'/ϵ in the effective Hamiltonian for the $K^0, \overline{K^0}$ system also give strong constraints on the amount of $\tilde{d}_L, \tilde{s}_L$ and $\tilde{d}_R, \tilde{s}_R$ mixing and CP violation in the soft terms.

Similarly:

The $D^0, \overline{D^0}$ system constrains $\tilde{u}_L, \tilde{c}_L$ and $\tilde{u}_R, \tilde{c}_R$ soft SUSY-breaking mixing.

The $B^0_d, \overline{B^0_d}$ system constrains $\tilde{d}_L, \tilde{b}_L$ and $\tilde{d}_R, \tilde{b}_R$ soft SUSY-breaking mixing.

In general, the soft-SUSY breaking terms must be either very heavy, or nearly flavor-blind, to avoid flavor-changing violating experimental limits.

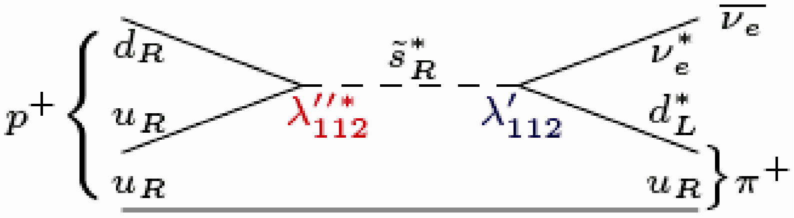
Actually, the most general possible superpotential would also include:

$$W_{\Delta L=1} = \frac{1}{2}\lambda_{ijk} L_i L_j \bar{e}_k + \lambda'_{ijk} L_i Q_j \bar{d}_k + \mu'_i L_i H_u$$

$$W_{\Delta B=1} = \frac{1}{2}\lambda''_{ijk} \bar{u}_i \bar{d}_j \bar{d}_k$$

These violate lepton number ($\Delta L = 1$) or baryon number ($\Delta B = 1$).

If both types of couplings were present, and of order 1, then the proton would decay in a tiny fraction of a second through diagrams like this:



Many other proton decay modes, and other experimental limits on B and L violation, give strong constraints on these terms in the superpotential.

One postulates a new discrete symmetry called **Matter Parity**, also known as **R-parity**.

R-parity is defined for each particle with spin S by:

$$P_R = (-1)^{3(B-L)+2S}$$

Consequences of R-Parity Conservation

The particles with odd R-parity ($P_R = -1$) are the "supersymmetric particles" or "sparticles".

Every interaction vertex in the theory must contain an even number of $P_R = -1$ sparticles. Three extremely important consequences:

- The lightest sparticle with $P_R = -1$, called the "Lightest Supersymmetric Particle" or LSP, must be absolutely stable. If the LSP is electrically neutral, it interacts only weakly with ordinary matter, and so can make an attractive candidate for the non-baryonic dark matter required by cosmology.
- In collider experiments, sparticles can only be produced in even numbers (usually two-at-a-time).
- Each sparticle other than the LSP must eventually decay into a state that contains an odd number of LSPs (usually just one). The LSP escapes the detector, with a missing momentum signature.

Backup

2-Component vs. 4-Component Formalism

Two LH Weyl spinors ξ, χ can form a 4-component Dirac or Majorana spinor:

$$\Psi = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}$$

In the 4-component formalism, the Dirac Lagrangian is:

$$\mathcal{L} \quad = \quad -i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi, \qquad \text{where} \qquad \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix},$$

In the two-component fermion language, with spinor indices suppressed:

$$\mathcal{L} = -i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi - i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi - m(\xi\chi + \xi^\dagger\chi^\dagger),$$

up to a total derivative. This follows from using the identity:

$$-i\chi\sigma^\mu\partial_\mu\chi^\dagger = -i\partial_\mu\chi^\dagger\overline{\sigma}^\mu\chi.$$

A Majorana fermion can be described in 4-component language in the same way by identifying $\chi = \xi$, and multiplying the Lagrangian by a factor of $\frac{1}{2}$ to compensate for the redundancy.

Example: Standard Model in 2-Component Notation

For example, to describe the Standard Model fermions in 2-component notation:

$$\begin{aligned}\mathcal{L} &= -iQ^{\dagger i}\overline{\sigma}^{\mu} D_{\mu} Q_i - i\bar{u}^{\dagger i}\overline{\sigma}^{\mu} D_{\mu}\bar{u}_i - i\bar{d}^{\dagger i}\overline{\sigma}^{\mu} D_{\mu}\bar{d}_i \\ &\quad -iL^{\dagger i}\overline{\sigma}^{\mu} D_{\mu} L_i - i\bar{e}^{\dagger i}\overline{\sigma}^{\mu} D_{\mu}\bar{e}_i\end{aligned}$$

with the family index $i = 1, 2, 3$ summed over, color and weak isospin and spinor indices suppressed, and D_{μ} the appropriate Standard Model covariant derivative, for example,

$$\begin{aligned}D_{\mu} L &= \left[\partial_{\mu} + i\frac{g}{2} W_{\mu}^{a}\tau^{a} - i\frac{g'}{2} B_{\mu}\right]\begin{pmatrix}\nu_e \\ e\end{pmatrix} \\ D_{\mu}\bar{e} &= \left[\partial_{\mu} + ig' B_{\mu}\right]\bar{e}\end{aligned}$$

with τ^{a} $(a = 1, 2, 3)$ equal to the Pauli matrices, and the gauge eigenstate weak bosons are related to the mass eigenstates by

$$\begin{aligned}W_{\mu}^{\pm} &= (W_{\mu}^{1} \mp W_{\mu}^{2})/\sqrt{2}, \\ \begin{pmatrix}Z_{\mu} \\ A_{\mu}\end{pmatrix} &= \begin{pmatrix}\cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W\end{pmatrix}\begin{pmatrix}W_{\mu}^{3} \\ B_{\mu}\end{pmatrix}.\end{aligned}$$

Notations: 2-Component (Weyl) Spinors

Metric tensor: $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$

Position, momentum four-vectors: $x^\mu = (t, \vec{x})$; $\quad p^\mu = (E, \vec{p})$

Left-handed (LH) two-component Weyl spinor: $\psi_\alpha \quad \alpha = 1, 2$

Right-handed (RH) two-component Weyl spinor: $\psi^\dagger_{\dot{\alpha}} \quad \dot{\alpha} = 1, 2$

The Hermitian conjugate of a left-handed Weyl spinor is a right-handed Weyl spinor, and vice versa:

$$(\psi_\alpha)^\dagger = (\psi^\dagger)_{\dot{\alpha}} \equiv \psi^\dagger_{\dot{\alpha}}$$

(Some other people call this $\overline{\psi}_{\dot{\alpha}}$)

Therefore, **all** spin-$1/2$ fermionic degrees of freedom in any theory can be defined in terms of a list of left-handed Weyl spinors, $\psi_{i\alpha}$ where i is a flavor index. With this convention, right-handed Weyl spinors always carry a dagger: $\psi^{\dagger i}_{\dot{\alpha}}$.

Notations: 2-Component (Weyl) Spinors

Products of spinors are defined as:

$$\psi\xi \equiv \psi_\alpha \xi_\beta \epsilon^{\beta\alpha} \qquad \text{and} \qquad \psi^\dagger \xi^\dagger \equiv \psi^\dagger_{\dot\alpha} \xi^\dagger_{\dot\beta} \epsilon^{\dot\alpha\dot\beta}$$

Since ψ and ξ are anti-commuting fields, the antisymmetry of $\epsilon^{\alpha\beta}$ implies:

$$\psi\xi = \xi\psi = (\psi^\dagger\xi^\dagger)^* = (\xi^\dagger\psi^\dagger)^*.$$

To make Lorentz-covariant quantities, define matrices $(\overline{\sigma}_\mu)^{\dot\alpha\beta}$ and $(\sigma_\mu)_{\alpha\dot\beta}$ with:

$$\overline{\sigma}_0 = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \overline{\sigma}_n = -\sigma_n = (\vec{\sigma})_n \quad (\text{for } n = 1, 2, 3).$$

Then the Lagrangian for an arbitrary collection of LH Weyl fermions ψ_i is:

$$\mathcal{L} = -i\psi^{\dagger i}\overline{\sigma}^\mu D_\mu \psi_i - \tfrac{1}{2} M^{ij}\psi_i\psi_j - \tfrac{1}{2} M_{ij}\psi^{\dagger i}\psi^{\dagger j}$$

where D_μ = covariant derivative, and the mass matrix M^{ij} is symmetric, with $M_{ij} \equiv (M^{ij})^*$.