Particle Physics II

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Exercise sheet II

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Nov 11, 2011

Please use $c = 3 \times 10^8\,\mathrm{m/s}$ and $\hbar c = 0,2\,\mathrm{GeV\,fm}$ for all numerical computations.

In-class exercises

Exercise 11 *Conservation of Total Angular Momentum*

Show that the Hamiltonian ($H \equiv p^0$) for the Dirac Equation does not commute with the orbital angular momentum operator:

$$[H, \overrightarrow{L}] = -i\gamma^0(\overrightarrow{\gamma} \times \overrightarrow{p}) \neq 0. \tag{1}$$

Show also that the Dirac Equation does not commute with the spin operator:

$$[H, \overrightarrow{S}] = i\gamma^0(\overrightarrow{\gamma} \times \overrightarrow{p}), \tag{2}$$

but as a consequence, the total angular momentum $\overrightarrow{J} = \overrightarrow{L} + \overrightarrow{S}$ is conserved.

Finally, show that any spinor is an eigenstate of the S^2 operator, and that this spin is consistent with the observation that electrons and positrons are spin-1/2 particles.

Hint: recall that the commutation relation between position and momentum operators is: $[x_k, p_l] = i\partial_{kl}$.

Exercise 12 *Completeness of Dirac Equation Solutions*

Given these four solutions to the Dirac Equation:

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{(p_x+ip_y)}{E+m} \end{pmatrix}, u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, v^{(1)} = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, v^{(2)} = -N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}, \tag{3}$$

with $N = \sqrt{E+m}$, show that the completeness relation holds for these spinors, namely that:

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + mc), \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - mc). \tag{4}$$

Exercise 13 *Spinor Transformations Under Parity*

Use the spinors for an electron and positron at rest to show that these are eigenstates of the parity operator. Show that the electron and positron have opposite parity.

Exercise 14 *Non-relativistic Limit of the Dirac Equation*

Consider this form of the Dirac Equation:

$$\begin{pmatrix} m & \vec{\sigma}\cdot\vec{P} \\ \vec{\sigma}\cdot\vec{P} & -m \end{pmatrix} \psi = i\partial_t\psi, \tag{5}$$

and a non-relativistic electron travelling at speed $v \ll 1$ represented by $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$ where ψ_A and ψ_B are two-component spinors.

(i) Show that ψ_A fulfills the Pauli equation:

$$(\frac{1}{2m}(\vec{P} + e\vec{A})^2 + \frac{e}{2m}\vec{\sigma} \cdot \vec{B} - eA^0)\psi_A = E_{kin}\psi_A, \tag{6}$$

where E_{kin} is the kinetic energy of the electron, and $A^\mu = (A_0, \vec{A})$ is the four-vector potential that satisfy $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E}_{electric} = -\partial_t \vec{A} - \vec{\nabla} A^0$.
Hints:

- You can assume $|eA^0| \ll m$ and $E_{kin} \ll m$.
- You can start with the time evolution of spinors: $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = e^{-iEt} \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, as well as the substitutions: $\vec{P} \to \vec{P} + e\vec{A}$, $E \to E + eA^0$ in eliminating ψ_B.

(ii) Derive the gyromagnetic ratio g of the electron. Note that the magnetic moment of the electron is related to its spin through:

$$\vec{\mu} \equiv -g\frac{e}{2m}\vec{S}. \tag{7}$$

Hint: You can make use of the following:

- $(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$, in the case that $[\vec{a}, \vec{\sigma}] = [\vec{b}, \vec{\sigma}] = 0$
- $\vec{P} = -i\vec{\nabla}$
- $\vec{\nabla} \times (\vec{A}\psi) + \vec{A} \times (\vec{\nabla}\psi) = (\vec{\nabla} \times \vec{A})\psi$.

Exercise 15 *Target Scattering Energy Threshold*
If a particle A, with energy E hits a particle B at rest, and produces n particles $C_1, C_2, ..., C_n$ with masses $m_1, m_2, ..., m_n$, what is the energy threshold (minimum incident energy E_{min}) for this process to occur (in terms of $m_A, m_B, m_1, ..., m_n$)?

Home exercises

Exercise 16 *Dirac Matrix Properties* **4 Points**

Show the following are true, using the Dirac Representation of γ-Matrices:

(i) $(\gamma^0)^2 = 1$

(ii) $(\gamma^k)^2 = -1$ (for $k = 1, 2, 3$)

(iii) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

Show that (iii) is also true for the Weyl representation.

What does γ^5 look like in the Dirac representation and the Weyl representation?

Exercise 17 *Orthogonality of Dirac Equation Solutions* **3 Points**

For the above solutions to the Dirac equation, show that $u^{(1)}$ and $u^{(2)}$ are orthogonal, as are $v^{(1)}$ and $v^{(2)}$. Show however, that $u^{(1)}$ and $v^{(1)}$ are not orthogonal.

Exercise 18 *Helicity in the Dirac Equation* **7 Points**

The helicity operator is given by $\hat{h} = \frac{\vec{\sigma}\cdot\vec{p}}{2|\vec{p}|}$. Is helicity conserved in the Dirac equation? Construct the eigenspinors $u^{(\pm)}$ of the helicity operator for the electron that have eigenvalues $\pm\frac{1}{2}$.

Hints:

- the eigenspinors for helicity are linear combinations of the spinors $u^{(1)}$ and $u^{(2)}$: $u^{(\pm)} = au^{(1)} + bu^{(2)}$.
- these spinors should be normalized (which provides an additional constraint that can be used)
- the substitutions $A = \frac{a\sqrt{E+m}}{p_z \pm |\vec{p}|}$ and using the bispinor $u = \begin{pmatrix} p_z \pm |\vec{p}| \\ p_x + ip_y \end{pmatrix}$ might be helpful.

Exercise 19 *Charge Conjugation and Time Reversal* **4 Points**

The charge conjugation operator C transforms spinors via: $\psi \to C(\psi) = \psi' = i\gamma^2\psi^*$. What do the charge conjugates of the spinors $v^{(1)}$ and $v^{(2)}$ look like?

The time reversal operator T transforms spinors via: $\psi \to T(\psi) = \psi' = i\gamma^1\gamma^3\psi^*$. What does the time reversed spinor of $u^{(1)}e^{-ip\cdot x}$ look like?

Exercise 20 *Adjunct Dirac Equation* **4 Points**

Given the Dirac Equation

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0, \tag{8}$$

show that the adjunct spinor $\bar{\psi} \equiv \psi^\dagger\gamma^0$ satisfies the adjunct Dirac Equation

$$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0. \tag{9}$$