# Particle Physics II

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# Exercise sheet III

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## In-class exercises

**Exercise 21** *QED: $e^+e^- \to \mu^+\mu^-$*

Calculate the *unpolarized*, non-resonant differential ($d\sigma/d\Omega$) and total cross section of the process $e^+e^- \to \mu^+\mu^-$ in the center-of-mass frame of reference. The following symbols are used: $p_1$, $p_2$, $p_3$, $p_4$, $q$... four-momenta of $e^-$, $e^+$, $\mu^-$, $\mu^+$, photon, in that order.

(i) Show that the matrix element (omitting spin indices) is given by

$$\mathfrak{M} = \frac{e^2}{q^2}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right] \tag{1}$$

(ii) Show that the square of the matrix element is given by

$$|\mathfrak{M}|^2 = \frac{e^4}{q^4}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu v(p_2)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\bar{v}(p_4)\gamma_\nu u(p_3)\right] \tag{2}$$

(iii) Show that the spin-averaged matrix element squared is given by

$$\frac{1}{4}\sum_{\text{spins}}|\mathfrak{M}|^2 = \frac{e^4}{4q^4}\text{tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right]\text{tr}\left[(\not{p}_3 + m_\mu)\gamma_\mu(\not{p}_4 - m_\mu)\gamma_\nu\right] \tag{3}$$

Hint: Write in spinor indices, and use the complete relations $\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m$, $\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m$.

(iv) You can now use the approximation $m_e \to 0$. Using the theorems for traces involving $\gamma$ matrices from the lecture, show that this expression can be simplified to

$$\frac{1}{4}\sum_{\text{spins}}|\mathfrak{M}|^2 = \frac{8e^4}{q^4}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_\mu^2(p_1\cdot p_2)\right] \tag{4}$$

(v) Show that in the center-of-mass frame of reference, this can be simplified to

$$\frac{1}{4}\sum_{\text{spins}}|\mathfrak{M}|^2 = e^4\left[1 + \frac{m_\mu^2}{E^2} + (1 - \frac{m_\mu^2}{E^2})\cos^2\theta\right] \tag{5}$$

where $\theta$ is the angle between $p_1$ and $p_3$, $E$ is the energy of any of the four fermions.

(vi) Using the Golden Rule for $2 \to 2$ scattering derived in exercise session 1, show that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + (1 - \frac{m_\mu^2}{E^2})\cos^2\theta\right] \tag{6}$$

where alpha is defined as $\alpha = e^2/(4\pi)$.

(vii) Integrate this expression to show that the total cross section is given by

$$\sigma = \frac{\pi\alpha^2}{3E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{E^2}\right) \tag{7}$$

Discussion of the result:
a) What minimum energy of the incoming fermions is required for this result to be valid?
b) Interprete the cross section in the high-energy limit ($E \gg m_\mu$), keeping only the leading terms.

# Home exercises

**Exercise 22** *Luminosity measurement at LEP* **7 Punkte**

The measurement of a cross section usually requires the knowledge of the luminosity $\mathcal{L}(t)$. The luminosity relates the recorded event rate and the cross section via

$$\frac{\mathrm{d}\dot{N}}{\mathrm{d}\Omega} = \mathcal{L}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}. \tag{8}$$

For a collider experiment, the luminosity is given as

$$\mathcal{L} = f N_b \frac{N_1 N_2}{4\pi\sigma_x\sigma_y}, \tag{9}$$

where $f$ is the orbital frequency of the particle beams, $N_b$ the number of particle bunches, $N_1$ and $N_2$ the number of particles in one bunch, and $\sigma_x$ and $\sigma_y$ the size of the transverse dimension of the beams at the interaction point.

LEP was an $e^+e^-$ collider with a circumsphere of about 27 km, running at a center-of-mass energy of $\sqrt{s} = 91.2$ GeV from 1989 to 1995 (in its final year 2000, LEP ran at up to 209 GeV). Both beams contained four bunches with $3 \times 10^{11}$ particles each. The beams were focused to $\sigma_x = 100\,\mu$m und $\sigma_y = 20\,\mu$m at the interaction points.

(i) What is the unit of luminosity?

(ii) Calculate the luminosity at LEP.

(iii) The cross section for $Z^0$ bosons at this energy is about 60 nb. How many such bosons have been produced in a typical year (data-taking for about $10^7$ s)?

(iv) Measuring the quantities entering the luminosity definition is associated with large uncertainties. For this reason, the LEP experiments measured the luminosity via a reference process, Bhabha scattering at small angles. This cross section is known to very good precision, and at small angles given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha_{\mathrm{em}}^2}{s}\frac{1}{\sin^4(\theta/2)}. \tag{10}$$

One of the four LEP experiments, OPAL, used special calorimeters (called luminosity monitors) to detect Bhabha scattering events. They were located along the beam pipe in a distance of 2.5 m from the interaction point in both directions and were sensitive in a radial distance to the beam axis of $77 - 127$ mm. How large is the cross section for Bhabha scattering in this sensitive region (the so-called detector acceptance)?

(v) In one year of data-taking, about 800 000 Bhabha events were recorded. At the same time, 17 128 $e^+e^- \to \mu^+\mu^-$ events were recorded. Assuming that 100% of all Bhabha events, and 90% of all muon pair production events were recorded — what is the LEP cross section for muon pair production at $\sqrt{s} = 91.2$ GeV?

(vi) Assume the inner distance of the luminosity monitors to the beam pipe is known to an accuracy of 1 mm. How large is the relative uncertainty induced in the luminosity measurement?

**Exercise 23** *Differential cross section for $e^-\mu^- \to e^-\mu^-$* **4 Punkte**

During the lecture, the differential cross section for the process $e^-\mu^- \to e^-\mu^-$ has been evaluated to be

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{s^2+u^2}{t^2}$$

in the high-energy limit ($E \gg m_\mu$, $E \gg m_e$).

Using the angular dependence of the Mandelstam variables, derive the differential cross section $\frac{d\sigma}{d\theta}$ for the scattering angle $\theta$. Show that this cross section has its maximum in the forward direction, and that the total cross section diverges.

Hint: You can use the same approximations as used for deriving $\frac{d\sigma}{d\Omega}$. You do not need to integrate over $\theta$ to show that the cross section diverges.

**Exercise 24** *$e^-\mu^-$ scattering in the muon rest frame* **5 Punkte**

During the lecture it was shown that the spin-average matrix element squared for the process $e^-\mu^- \to e^-\mu^-$ can be written as

$$\frac{1}{4}\sum_{\text{spins}} |\mathfrak{M}|^2 = \frac{8e^4}{t^2}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - M^2(p_1 \cdot p_3) - m^2(p_2 \cdot p_4) + 2m^2M^2\right]$$

where $t = (p_1 - p_3)^2$, $M$ is the mass of the muon, and $m$ the mass of the electron. $p_1$ and $p_2$ are the four-momenta of the incoming electron and muon (in that order), and $p_3$ and $p_4$ of the outgoing electron and muon (in that order).

(i) Show that when using the approximation $m = 0$, this term can be written as

$$\frac{1}{4}\sum_{\text{spins}} |\mathfrak{M}|^2 = \frac{8e^4}{t^2}\left[-\frac{1}{2}t(p_1 \cdot p_2 - p_2 \cdot p_3) + 2(p_1 \cdot p_2)(p_2 \cdot p_3) + \frac{1}{2}M^2 t\right].$$

(ii) Show that in the muon rest frame (i.e., for an initial-state muon at rest) this expression can be evaluated to

$$\frac{1}{4}\sum_{\text{spins}} |\mathfrak{M}|^2 = \frac{16e^4}{t^2}M^2E_3E_1\left[\cos^2\frac{\theta}{2} - \frac{t}{2M^2}\sin^2\frac{\theta}{2}\right].$$

where $\theta$ is the angle between incoming and outgoing electron.
Hint: Express $t$ as a function of $\sin^2\frac{\theta}{2}$.

**Exercise 25** *Helicity and chirality* **4 Punkte**

(i) For the solution of the Dirac equation

$$u(p) = \sqrt{E+m}\begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \end{pmatrix} \quad \text{with } \chi = (1,0),$$

show that: for the case of a massless particle, applying the helicity operator

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{p} = \frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}$$

is equal to applying the chirality operator

$$\frac{1}{2}\gamma^5 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that this result is also a good approximation for a massive particle in the high-energy limit, $E \gg m \to E \simeq p$.

(ii) The chirality projection operators $P_L = \frac{1}{2}(\mathbf{1} - \gamma^5)$ and $P_R = \frac{1}{2}(\mathbf{1} + \gamma^5)$ define the chiral states $u_{L,R}$ (called "left-handed" and "right-handed" states) as $u_L \equiv P_L u$ and $u_R \equiv P_R u$. Show that

$$P_L u_L = u_L,$$

$$P_R u_R = u_R,$$

$$P_L u_R = P_R u_L = 0.$$

(iii) Assume that a spinor $u$ can be written as a sum of its left- and right-handed components, $u = u_L + u_R$. Then a similar relation holds for $\bar{u}$. Show that the following equation is valid:

$$\bar{u}\gamma^\mu u = \bar{u}_R\gamma^\mu u_R + \bar{u}_L\gamma^\mu u_L,$$

This implies that chirality is conserved in each vertex; and thus also helicity for the case of massless particles.