Particle Physics II

Markus Schumacher

Exercise sheet IV

Martin Flechl and Stan Lai

Nov 25, 2011

In-class exercises

Exercise 26 *QED (continued):* $e^+e^- \to \mu^+\mu^-$

Last week, it was shown that the unpolarized amplitude squared for $e^+e^- \to \mu^+\mu^-$ was:

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m_\mu^2 (p_1 \cdot p_2)\right]$$

in the limit $m_e \to 0$. Here, p_1, p_2, p_3, p_4, q are the four-momenta of the e^-, e^+, μ^-, μ^+, and virtual photon.

(i) Show that in the center-of-mass frame of reference, this can be simplified to

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = e^4\left[1 + \frac{m_\mu^2}{E^2} + (1 - \frac{m_\mu^2}{E^2})\cos^2\theta\right] \tag{1}$$

where θ is the angle between p_1 and p_3, E is the energy of any of the four fermions.

(ii) Using the Golden Rule for $2 \to 2$ scattering derived in exercise session 1, show that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + (1 - \frac{m_\mu^2}{E^2})\cos^2\theta\right] \tag{2}$$

where alpha is defined as $\alpha = e^2/(4\pi)$.

(iii) Integrate this expression to show that the total cross section is given by

$$\sigma = \frac{\pi\alpha^2}{3E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{m_\mu^2}{2E^2}\right) \tag{3}$$

Discussion of the result:
a) What minimum energy of the incoming fermions is required for this result to be valid?
b) Interpret the cross section in the high-energy limit ($E \gg m_\mu$), keeping only the leading terms.

Exercise 27 *Modified QED: Pair Annihilation*

Imagine that the photon (force carrier for QED), instead of being a massless spin 1 particle, was a spin 0 particle of with mass m_γ. In this case:

- The vertex factor changes from $iQe\gamma^\mu$ to iQe,
- The photon propagator changes from $\frac{-ig_{\mu\nu}}{q^2}$ to $\frac{-i}{q^2 - m_\gamma^2}$.
- No photon polarization factors ϵ_μ are needed for external photons.

(i) Calculate the scattering amplitude $\mathcal{M}$ for $e^+e^- \to \gamma\gamma$ in this theory.

(ii) Determine the $|\mathcal{M}|^2$ in the relativistic limit ($m_e, m_\gamma \to 0$), by averaging over intial state spins.

Home exercises

Exercise 28 *Electron-Positron Annihilation* **6 Points**

(i) From the Feynman rules of QED, derive the scattering amplitudes $\mathcal{M}$ for the t-channel and u-channel processes at leading order for the process $e^+e^- \to \gamma\gamma$.

(ii) Assume that only the t-channel contributes to this process. Calculate the the quantity $|\mathcal{M}|^2$ in the relativistic limit ($m_e = 0$), averaging over initial spins, and summing over final polarizations. Show that in terms of Mandelstam variables, that

$$|\mathcal{M}|^2 = 2e^4 \frac{t}{u}.$$

Hint:
Note that you can use the following substitution, when calculating the amplitude:

$$\sum_{polarizations} \epsilon_\mu^{(s)} \epsilon_\nu^{(s)*} = -g_{\mu\nu}$$

Exercise 29 *Compton Scattering* **7 Points**

Compton Scattering describes the process $e^-\gamma \to e^-\gamma$.

(i) From the Feynman rules of QED, obtain the amplitudes for the two leading order diagrams.

(ii) Using crossing symmetry and Mandelstam variables, relate the diagrams in the two Compton Scattering scattering amplitudes to those from Electron-Positron Annihilation.

(iii) Given that $|\mathcal{M}|^2 = 2e^4(\frac{u}{t} + \frac{t}{u})$ for Electron-Positron Annihilation into a pair of photons, what is the corresponding $|\mathcal{M}|^2$ for Compton Scattering?

(iv) Show that the differential cross-section for Compton Scattering in the rest frame of the target electron is given by the Klein-Nishina formula:

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right], \tag{4}$$

where ω and ω' are the frequencies of the incident and scattered photons, and θ is the angle between the outgoing and incoming photon momenta.
Use the fact that the unpolarized scattering amplitude squared can be written as:

$$|\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right],$$

where p, k, k' are the four-momenta of the initial electron, initial photon, and scattered photon, respectively. This can be related to the differential cross-section using:

$$\frac{d\sigma}{d(\cos\theta)} = \frac{(\omega')^2}{32\pi\omega^2 m^2}(|\mathcal{M}|^2).$$

Hint:
Show that

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta)$$

is true in the rest frame of the electron.

Exercise 30 *Modified QED: Decays* **7 Points**

Recall the Feynman rules for modified QED with a scalar, massive photon:

- The vertex factor changes from $iQe\gamma^\mu$ to iQe,
- The photon propagator changes from $\frac{-ig_{\mu\nu}}{q^2}$ to $\frac{-i}{q^2-m_\gamma^2}$.
- No photon polarization factors ϵ_μ are needed for external photons.

(i) Calculate the scattering amplitude for the process $\gamma \to e^+e^-$.

(ii) Show that summing over final state spins yields:

$$|\mathcal{M}|^2 = 2e^2(m_\gamma^2 - 4m_e^2).$$

(iii) Show that the decay rate is given by

$$\Gamma = \frac{e^2}{8\pi m_\gamma^2}(m_\gamma^2 - 4m_e^2)^{3/2}.$$

(iv) Calculate the lifetime of the photon if $m_\gamma = 10$ MeV. Calculate the lifetime if $m_\gamma = 1$ GeV.