Particle Physics II

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Problem Set II

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In-class exercises

Exercise 8 *Conservation of Total Angular Momentum*

(a) The Hamiltonian in relativistic quantum mechanics is just $H \equiv p^0$. Write the Hamiltonian H in terms of 3-momentum and mass operators, by rearranging the Dirac equation.
(Remember the quantum mechanical momentum operator for momentum is given by $p_\mu = i\partial_\mu$.)

(b) Show that the Hamiltonian ($H \equiv p^0$) for the Dirac Equation does not commute with the orbital angular momentum operator:

$$[H, \overrightarrow{L}] = -i\gamma^0(\overrightarrow{\gamma} \times \overrightarrow{p}) \neq 0.$$

(Since $\vec{L} = \vec{r} \times \vec{p}$, we can write $L^i = \epsilon^{ijk} r_j p_k$.)

(c) Show also that this Hamiltonian does not commute with the spin operator:

$$[H, \overrightarrow{S}] = i\gamma^0(\overrightarrow{\gamma} \times \overrightarrow{p}).$$

(Recall that $\vec{S} = \vec{\Sigma}/2$ whereby $\vec{\Sigma}$ are just the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$.)

(d) However, show that as a consequence, the total angular momentum $\overrightarrow{J} = \overrightarrow{L} + \overrightarrow{S}$ is conserved.

Hint: recall that the commutation relation between position and momentum operators is: $[x_k, p_l] = i\partial_{kl}$.

Exercise 9 *Completeness of Dirac Equation Solutions*

Given these four solutions to the Dirac Equation:

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, v^{(1)} = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, v^{(2)} = -N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix},$$

with $N = \sqrt{E+m}$, show that the completeness relation holds for these spinors, namely that:

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + m), \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - m).$$

Exercise 10 *Adjunct Dirac Equation*

Given the Dirac Equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0,$$

show that the adjunct spinor $\bar{\psi} \equiv \psi^\dagger \gamma^0$ satisfies the adjunct Dirac Equation

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0.$$

Exercise 11 *Target Scattering Energy Threshold*

If a particle A, with energy E hits a particle B at rest, and produces n particles $C_1, C_2, ..., C_n$ with masses $m_1, m_2, ..., m_n$, what is the energy threshold (minimum incident energy E_{min}) for this process to occur (in terms of $m_A, m_B, m_1, ..., m_n$)?

Homework

Exercise 12 *Dirac and Klein-Gordon Equations* **4 Points**

Show that a spinor ψ satisfying the Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$ also satisfies the Klein-Gordon equation $(\partial^\mu \partial_\mu + m^2)\psi = 0$.
Hint: Operate on the left for both sides of the Dirac equation with the operator $\gamma^\nu \partial_\nu$.

Exercise 13 *Dirac Matrix Properties* **4 Points**

Show the following are true, using the Dirac Representation of γ-Matrices:

(a) $(\gamma^0)^2 = 1$
(b) $(\gamma^k)^2 = -1$ (for $k = 1, 2, 3$)
(c) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

Show that (c) is also true for the Weyl representation.

What does γ^5 look like in the Dirac representation and the Weyl representation?

Exercise 14 *Orthogonality of Dirac Equation Solutions* **4 Points**

For the above solutions to the Dirac equation, show that $u^{(1)}$ and $u^{(2)}$ are orthogonal, as are $v^{(1)}$ and $v^{(2)}$. Show however, that $u^{(1)}$ and $v^{(1)}$ are not orthogonal.
Hint: Show for instance that $u^{(1)\dagger} u^{(2)} = 0$.

Exercise 15 *Helicity in the Dirac Equation* **4 Points**

The helicity operator is given by $\hat{h} = \frac{\vec{\sigma} \cdot \vec{p}}{2|\vec{p}|}$. Show that helicity is conserved in the Dirac equation.
Hint: Show that the helicity operator commutes with the Hamiltonian $H \equiv p^0$.

Exercise 16 *Charge Conjugation and Time Reversal* **4 Points**

The charge conjugation operator C transforms spinors via: $\psi \to C(\psi) = \psi' = i\gamma^2 \psi^*$. What do the charge conjugates of the spinors $v^{(1)}$ and $v^{(2)}$ look like?

The time reversal operator T transforms spinors via: $\psi \to T(\psi) = \psi' = i\gamma^1 \gamma^3 \psi^*$. What does the time reversed spinor of $u^{(1)} e^{-ip \cdot x}$ look like?