Particle Physics II

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Problem Set III

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In-class exercises

Exercise 17 *Non-relativistic Limit of the Dirac Equation*

(a) Show that the Dirac equation can also be written in this form:

$$\begin{pmatrix} m & \vec{\sigma}\cdot\vec{P} \\ \vec{\sigma}\cdot\vec{P} & -m \end{pmatrix}\psi = i\partial_t\psi,$$

where ψ_A and ψ_B are two-component spinors, and $\vec{P}$ is the momentum operator.
Hint: Use the Dirac representation of the γ-matrices.

(b) In classical mechanics, it can be shown that a charged particle with charge e in the presence of a Lorentz force satisfies the following relations for the scalar and vector potentials $A^0 = U(\vec{x},t)$ and $\vec{A}(\vec{x},t)$:

$$\vec{p} = m\frac{d\vec{x}}{dt} + e\vec{A}(\vec{x},t) \quad \text{and} \quad H = \frac{1}{2m}\left[\vec{p} - e\vec{A}(\vec{x},t)\right]^2 + eU(\vec{x},t)$$

for the mechanical momentum and the Hamiltonian, respectively. In Einstein notation, The scalar and vector potentials are written as a 4-component vector object: $A^\mu = (U,\vec{A})$. Thus, the dynamics of a spin-1/2 charged particle interacting with a classical vector field can be expressed by the Dirac equation in part (a), with the substitutions:

$$\vec{P} \to \vec{P} + e\vec{A} \quad \text{and} \quad E \to E + eA^0.$$

Using the time evolution of the two-component spinors (E_{kin} being the kinetic energy of the electron):

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = e^{-i(E_{kin}+m)t}\begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

show that ψ_B can be written in terms of ψ_A as:

$$\psi_B \simeq \frac{\vec{\sigma}\cdot(\vec{P}+e\vec{A})}{2m}\psi_A.$$

Note you can make use of the non-relativistic approximations that $|eA^0| \ll m$ and $E_{kin} \ll m$.

(c) Substituting this expression for ψ_B, show that one arrives finally at the Pauli equation:

$$(\frac{1}{2m}(\vec{P}+e\vec{A})^2 + \frac{e}{2m}\vec{\sigma}\cdot\vec{B} - eA^0)\psi_A = E_{kin}\psi_A,$$

where the four-vector potential satisfies $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E}_{electric} = -\partial_t\vec{A} - \vec{\nabla}A^0$.
Hint: You can make use of the following:

- $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$, in the case that $[\vec{a},\vec{\sigma}] = [\vec{b},\vec{\sigma}] = 0$
- $\vec{P} = -i\vec{\nabla}$
- $\vec{\nabla}\times(\vec{A}\psi) + \vec{A}\times(\vec{\nabla}\psi) = (\vec{\nabla}\times\vec{A})\psi.$

Homework

Exercise 18 *Target Scattering Energy Threshold* **3 Points**

If a particle A, with energy E hits a particle B at rest, and produces n particles $C_1, C_2, ..., C_n$ with masses $m_1, m_2, ..., m_n$, what is the energy threshold (minimum incident energy E_{min}) for this process to occur (in terms of $m_A, m_B, m_1, ..., m_n$)?

Exercise 19 *Adjoint Dirac Equation* **3 Points**

Given the Dirac Equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0,$$

show that the adjoint spinor $\bar{\psi} \equiv \psi^\dagger \gamma^0$ satisfies the adjoint Dirac Equation

$$i\partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} = 0.$$

Exercise 20 *Transformations of Bilinear Covariants* **6 Points**

Recall from the lectures that the transformation matrix for spinors for a Lorentz-Boost in the z-direction is represented by:

$$S_{\text{Lor}} = \mathbf{1}_4 \cosh\frac{\omega}{2} - \gamma^0\gamma^3 \sinh\frac{\omega}{2}$$

and

$$S_{\text{Lor}}^{-1} = \mathbf{1}_4 \cosh\frac{\omega}{2} + \gamma^0\gamma^3 \sinh\frac{\omega}{2},$$

where $\cosh\omega = \gamma = \frac{E}{m}$, $\sinh\omega = \beta\gamma = \frac{|\vec{p}|}{m}$. The analagous transformation matrix for spinors for a rotation in space is represented by:

$$S_{\text{Rot}} = \exp\left(-\frac{\theta}{2}\gamma^1\gamma^2\right) = \mathbf{1}_4 \cos\frac{\theta}{2} - \gamma^1\gamma^2 \sin\frac{\theta}{2}$$

and

$$S_{\text{Rot}}^{-1} = \exp\left(+\frac{\theta}{2}\gamma^1\gamma^2\right) = \mathbf{1}_4 \cos\frac{\theta}{2} + \gamma^1\gamma^2 \sin\frac{\theta}{2}.$$

(a) Show the invariance of the pseudoscalar bilinear $p = \bar{\psi}\gamma^5\psi$ under a Lorentz-Boost as well as under a rotation about an angle θ.

(b) Determine the transformation for the axialvector bilinear $k^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$ under a Lorentz-Boost as well as under a rotation about an angle θ.

The following identities might help:

- $\gamma^0 S^\dagger \gamma^0 = S^{-1}$
- $\Lambda^\nu{}_\mu \gamma^\mu = S^{-1}\gamma^\nu S$ for the standard Lorentz transformation matrix Λ.

Exercise 21 *Gyromagnetic Ratio of the Electron* **3 Points**

Using the results from Problem 17(c), derive the gyromagnetic ratio g of the electron. Note that the magnetic moment of the electron is related to its spin through:

$$\vec{\mu} \equiv -g\frac{e}{2m}\vec{S}.$$

Hint: What is the potential energy produced by an external magnetic field $\vec{B}$ in terms of the magnetic moment $\vec{\mu}$?

Exercise 22 *Helicity and chirality* **5 Points**

(a) For the solution of the Dirac equation

$$u(p) = \sqrt{E+m}\begin{pmatrix}\chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi\end{pmatrix} \quad \text{with } \chi = (1,0),$$

show that for the case of a massless particle, applying the helicity operator

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{p} = \frac{1}{2}\begin{pmatrix}\vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p}\end{pmatrix}$$

is equal to applying the chirality operator

$$\frac{1}{2}\gamma^5 = \frac{1}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}.$$

Note that this result is also a good approximation for a massive particle in the high-energy limit, $E \gg m \rightarrow E \simeq p$.

(b) The chirality projection operators $P_L = \frac{1}{2}(\mathbf{1}-\gamma^5)$ and $P_R = \frac{1}{2}(\mathbf{1}+\gamma^5)$ define the chiral states $u_{L,R}$ (called "left-handed" and "right-handed" states) as $u_L \equiv P_L u$ and $u_R \equiv P_R u$. Show that

$$P_L u_L = u_L,$$

$$P_R u_R = u_R,$$

$$P_L u_R = P_R u_L = 0.$$

(c) Assume that a spinor u can be written as a sum of its left- and right-handed components, $u = u_L + u_R$. Then a similar relation holds for $\bar{u}$. Show that the following equation is valid:

$$\bar{u}\gamma^\mu u = \bar{u}_R\gamma^\mu u_R + \bar{u}_L\gamma^\mu u_L,$$

This implies that chirality is conserved in each vertex; and thus also helicity for the case of massless particles.