Particle Physics II

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Problem Set VIII

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In-class exercises

Exercise 41 *Equations of Motion for the Photon Field*

The free Lagrangian for the photon field is given by:

$$\mathcal{L}_{\text{free}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

From the Euler-Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu}$$

show that the equations of motion for the photon field are:

$$\partial_\mu F^{\mu\nu} = 0.$$

Show that Maxwell's equations $\nabla \cdot \vec{E} = 0$ and $-\frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = 0$ (in the absence of charges) are reproduced when interpreting $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

Exercise 42 *Quark-Antiquark Scattering and Colour Factors*

The expression for the matrix element (scattering amplitude) for quark-antiquark scattering is given by:

$$\mathcal{M} = \frac{-g_s^2}{q^2} \left[\bar{u}(3)\gamma^\mu u(1)\right] \left[\bar{v}(2)\gamma_\mu v(4)\right] \frac{1}{4} \left[c_3^\dagger \lambda^a c_1\right] \left[c_2^\dagger \lambda^a c_4\right].$$

Let's look at this expression in a bit more detail, to see what we can learn about this.

(a) Explain in words what information the following content contains in the calculation of the matrix element for the following objects in the equation:
- g_s
- q^2
- $\bar{u}(3), u(1), \bar{v}(2), v(4)$
- $c_3^\dagger, c_1, c_2^\dagger, c_4$

(b) What dimensions do the following objects have?
- $\bar{u}(3), \bar{v}(2)$
- $u(1), v(4)$
- $c_3^\dagger, c_2^\dagger$
- c_1, c_4
- γ^μ
- λ^a

The matrix element is typically written this way, since the calculation for the left hand side

$$\frac{-g_s^2}{q^2} \left[\bar{u}(3)\gamma^\mu u(1)\right] \left[\bar{v}(2)\gamma_\mu v(4)\right]$$

is the same calculation as for electron-positron scattering in QED. The right hand side of the expression:

$$f = \frac{1}{4} \left[c_3^\dagger \lambda^a c_1\right] \left[c_2^\dagger \lambda^a c_4\right]$$

is often called the colour factor, which is directly proportional to the matrix element.

(c) Let's consider that we have in the initial state, the quark and antiquark in the colour octet configuration. One example of this is that the incoming quark possesses the colour red, and the antiquark has the colour antiblue. What would be the colour of the outgoing quark and antiquark? Calculate the colour factor f in this particular case.

Homework

Exercise 43 *Equations of Motion for the QCD Lagrangian* **4 Points**

A reminder here is given for the QCD Lagrangian that was discussed in the lectures:

$$\mathcal{L}_{QCD} = \bar{\psi}(i\gamma_\mu D^\mu - m)\psi - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a.$$

The quark fields are given by ψ (don't forget they have spin and colour degrees of freedom), and $D^\mu = \partial^\mu - ig_s G^\mu$ for the gauge (gluon) fields $G^\mu = \frac{1}{2}\lambda^a G^\mu_a$, while $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + gf_{abc}G^b_\mu G^c_\nu$ is used for the free-term for the gluon fields. The variable g_s sets the strong interaction coupling strength, analogous to e in QED, while f_{abc} are the structure constants of QCD which satisfy $\left[\lambda^a, \lambda^b\right] = 2if^{abc}\lambda^c$ for Gell-Mann matrices λ^a. Remember the latin here indices go over the values 1...8 (the Gell-Mann matrices).

(a) Using the Euler-Lagrange equations derive the equations of motion for the quark fields (but not the gluon fields).

(b) Show that the 3-gluon-vertex and 4-gluon-vertex interactions scale with g_s and g_s^2 respectively, given this Lagrangian and write down the terms in the Lagrangian that correspond to these interactions.

Exercise 44 *Parity Conservation of Lagrangians* **5 Points**

Show that the Lagrangians for both QED and QCD are invariant under the parity transformation. Note that you can make use of bilinear covariants introduced for spinors, which can simplify the proof.

Exercise 45 *Interactions between quarks* **11 Points**

Let's consider first the interactions of two quarks.

(a) Draw the Feynman diagram for this process.

(b) The configuration of initial state quarks can be in the antisymmetric triplet combinations or in the symmetric sextet combinations. Write down the 3 and 6 combinations of the triplet and sextet configurations respectively, explicitly in terms of colour.

(c) For triplet and sextet configurations, calculate the colour factor (for only one example initial state in each configuration).

Now let's consider the interaction of three quarks together. The colour configurations can be grouped as follows:

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

a symmetric decuplet (10), two octets (8) in mixed symmetry, and an antisymmetric singlet (1).

Let's proceed to calculate the colour factor for the singlet state. Multi-particle interactions will be neglected here, such that the interaction between the three quarks will be broken down into the interaction of two quarks and a third observer/spectator. The sum of these combinations will give the final "colour factor" for the three-quark-state.

(d) The singlet state can be written:

$$\frac{1}{\sqrt{6}}\left(|RGB> - |GRB> + |GBR> - |BGR> + |BRG> - |RBG>\right).$$

Factor out the first particle, so you have terms such as $|R> \otimes(|GB> + |BG>)$ in the expression and determine the colour factor of each contribution from the "spectator quark" and the interacting quark pair for each term. Add these up to get the colour factor of the original quantum state.

(e) There are 10 possible combinations in the decuplet configuration. One possibility is when all quarks carry the exact same colour. What is the colour factor in this case? Another possibility in the decuplet configuration is

$$\frac{1}{\sqrt{3}}\left(|RRG> + |RGR> + |GRR>\right).$$

What's the colour factor for this combination?

(f) Does it make sense that nature chooses the singlet state rather than the decuplet state for stable baryons?